Semestral Examination

Algebra IV

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Instructions. $\mathbb{Q}$ denotes the field of rational numbers and a primitive nth root of unity is denoted by ζ_n.

All questions carry equal marks.

1. Let F be the field obtained by attaching a root of the polynomial $X^3 + X + 1$ to the field $K = \mathbb{Q}(\zeta_3)$. Show that $[F : K] = 3$ and that there is no $\alpha \in K$ such that $F = K(\sqrt[3]{\alpha})$.

2. Let p be a prime number and let $\mathbb{F}_p$ denote the finite field of order p. Prove that the field $\mathbb{F}_p(t, u)$ is not generated by a single element over its subfield $\mathbb{F}_p(t^p, u^p)$. (i.e. the Primitive Element Theorem does not hold here!)

3. Let K be a field and G be a finite subgroup of the group of field automorphisms of K. Prove that K is a finite Galois extension of the fixed field K^G with Galois group G.

4. Determine which roots of unity lie in each of the following fields: $\mathbb{Q}(\zeta_7)$, $\mathbb{Q}(\sqrt{-3})$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt[3]{-5})$.

5. Let K/k be a Galois extension, and let F be an intermediate field between k and K. Let H be the subgroup of $Gal(K/k)$ mapping F into itself. Show that H is the normalizer of $Gal(K/F)$ in $Gal(K/k)$.

6. Let K/k be a Galois extension with Galois group $\mathcal{S}_n$, the permutation group on n letters. Prove that K is the splitting field of an irreducible degree n polynomial in $k[X]$.